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A Class of Projective Modules which are Nearly Free

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Let R be a commutative ring with unit and consider the class of finitely generated projective modules P over the polynomial ring $R[x]$ which satisfy

$$R[x, x^{-1}] \otimes_{R[x]} P \text{ is free.} \tag{1}$$

By an obvious localization argument as in [6] it is immediate that (1) is equivalent to

$$x^n F \subset P \subset F \text{ with } F \text{ free}, \qquad n \in \mathbb{Z}. \tag{2}$$

The obvious question is whether all such modules are free. If this were so, it would be a major step toward the solution of Serre's problem. Suppose it is known that projective modules over a polynomial ring in $n - 1$ variables over a field are free. Let P be finitely generated and projective over $k[x_1, \ldots, x_n]$. Then P becomes free over $k(x_1)[x_2, \ldots, x_n]$ and, as above, we deduce that

$$f(x_1)F \subset P \subset F \text{ with } F \text{ free}, \qquad f \in k[x_1]. \tag{3}$$

By [5, Proof of 1.4], it will suffice to consider the case where f is a power of an irreducible polynomial. Now if k is algebraically closed, $f = (x_1 - a)^n$. Writing $x = x_1 - a$ we obtain (2).

In view of the known difficulty of Serre's problem, it seems unlikely that all projective modules satisfying (2) will be free. Nevertheless, we know of no counterexamples. In the positive direction, it is known that any P satisfying (2) is stably free [4, Theorem 1.3] [9, Theorem 5.3] and that P is free if $n = 1$ [9, Theorem 5.4]. We will show here that P is free if $n = 2$ and F has rank 2. This may be used in place of Lemma 1.3 in [5]. The proof of Lemma 1.3 in [5] does not give as much as our result. However, our proof uses very heavily the fact that $n = 2$ and rank $F = 2$. Therefore the proof in [5] is more likely to generalize than the present one.

Our theorem has a very bizarre consequence. Namely, that every unimodular row of the form (a^2, b, c) over a commutative ring can be completed to a unimodular matrix. Therefore, the projective module defined by such a row is free. Surprisingly enough, the analogous statement for longer rows is false.

1. The Main Theorem

As usual, we say that an $R[x]$-module F is extended if $F \approx R[x] \otimes_R F_0$. Clearly, $F_0 \approx F/xF$ and F is finitely generated and projective over $R[x]$ if and only if F_0 is so over R. A free module is obviously extended.

If P is a finitely generated projective A-module we let $\mathrm{rk}_{\mathfrak{p}}\, P$ be the rank of the free $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ and define $\mathrm{rk}\, P = \max \mathrm{rk}_{\mathfrak{p}}\, P$ over all prime ideals $\mathfrak{p}$ of A.

Theorem 1.1. *Let R be a commutative ring with unit. Let F and P be finitely generated projective $R[x]$-modules such that F is extended. Suppose $x^2F \subset P \subset F$ and* $\mathrm{rk}\, F \leqslant 2$. *Then $P \approx F$.*

Proof. If I is a nil ideal of $A = R[x]$ and $P/IP \approx F/IF$, then $P \approx F$ by a projective cover argument [1, Chap. III, 2.12, 2.4]. Let $J = \mathrm{nil}\, R$, $I = AJ$, $\bar{R} = R/I$, $\bar{P} = P/IP$, etc. Clearly $\mathrm{rk}\, \bar{F} \leqslant 2$. By [9, Cor. 5.2], F/P and P/x^2F are projective over R. Therefore $0 \to P \to F \to F/P \to 0$ and $0 \to x^2F \to P \to P/x^2F \to 0$ remain exact after reducing mod J so $x^2\bar{F} \subset \bar{P} \subset \bar{F}$. If $F = A \otimes_R F_0$, then $\bar{F} = \bar{A} \otimes_{\bar{R}} \bar{F}_0$ where $\bar{F}_0 = F_0/JF_0$. Thus all hypotheses of Theorem 1.1 are preserved by factoring out I so it will suffice to prove Theorem 1.1 under the additional hypothesis that R is reduced.

Now F/P and F/x^2F are finitely generated and projective by [9, Cor. 5.2]. Therefore [8, p. 143] we can write $\mathrm{Spec}\, R = U_1 \cup \cdots \cup U_r$, a disjoint union, such that F/P and F/x^2F have constant rank on each U_i. If $U_i = \mathrm{Spec}\, R_i$, this means that $R = R_1 \times \cdots \times R_r$ and $R_i \otimes_R F/P$ and $R_i \otimes_R F/x^2F$ have constant rank. Since the hypotheses and conclusion of Theorem 1.1 are preserved by applying $R_i \otimes_R -$, it is enough to prove the theorem with the additional hypothesis that F/P and F/x^2F have constant rank.

Let $r = \mathrm{rk}\, F/x^2F$ and $s = \mathrm{rk}\, F/P$. Then P/x^2F has contsant rank $t = r - s$. If $s = 0$ then $P = F$. If $t = 0$ then $P = x^2F \approx F$. Suppose that $s = 1$. By localizing, we see that the canonical map $R \to \mathrm{Hom}_R(F/P, F/P)$ is an isomorphism. Therefore x acts on F/P by multiplication by some $a \in R$. Since $x^2(F/P) = 0$, we have $a^2 = 0$. But R is reduced so $a = 0$. Therefore $x(F/P) = 0$ so $xF \subset P \subset F$ and we can apply [9, Theorem 5.4]. If $t = 1$, the same argument shows that $x(P/x^2F) = 0$ so $x^3F \subset xP \subset x^2F$ and [9, Theorem 5.4] again applies. Thus we can assume s, $t \geqslant 2$. Now F is induced so $F = A \otimes_R F_0$ and $\mathrm{rk}_A F = \mathrm{rk}_R F_0 \leqslant 2$. But $F/x^2F = F_0 \oplus xF_0$ so $\mathrm{rk}\, F_0 = \frac{1}{2}r$. Therefore $r \leqslant 4$. Since $r = s + t$ we must have $s = t = 2$ and $r = 4$. In this case, our argument will give a slightly more general result.

THEOREM 1.2. *Let R be a commutative ring with unit. Let F and P be finitely generated projective $R[x]$-modules such that F is extended. Suppose $x^2F \subset P \subset F$ and $\mathrm{rk}_R(P/x^2F) \leqslant 2$. Then $P \approx F$.*

Before proving this, we will consider the general situation where F and P are finitely generated projective $R[x]$-modules, F is extended, and $x^nF \subset P \subset F$. Our first objective is to obtain a presentation for P analogous to the matrix presentation used in [5].

Let $F = R[x] \otimes_R F_0$ and let $F_1 = F_0 \oplus F_0x \oplus \cdots \oplus F_0x^{n-1}$. Then $F = F_1 \oplus x^nF$ and $P = P_1 \oplus x^nF$ where $P_1 = F_1 \cap P$. If $p \in P_1$, then $xp \in P = P_1 \oplus x^nF$. Since $p \in F_1$, $xp \in F_1 \oplus x^nF_0$. Therefore we can write $xp = \alpha(p) + x^n\beta(p)$, where $\alpha: P_1 \to P_1$ and $\beta: P_1 \to F_0$ are R-homomorphisms. Note that if we identify P_1 with P/x^nF, it becomes an $R[x]$-module and the action of x is given by α. Therefore $\alpha^n = 0$. Now α and β induce maps $\alpha: P_1[x] \to P_1[x]$ and $\beta: P_1[x] \to F_0[x]$ where, as usual, $P_1[x] = R[x] \otimes_R P_1$, etc. The inclusion $i: P_1 \to P$ extends to an $R[x]$-homomorphism $j: P_1[x] \to P$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1[x] \arrow[r, "x-\alpha"] \arrow[d, "\beta"'] & P_1[x] \arrow[r] \arrow[d, "j"] & P_1 \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & F_0[x] \arrow[r, "x^n"] & P \arrow[r] & P/x^nF \arrow[r] & 0
\end{tikzcd}$$

commutes and the rows are exact. The top row is just the characteristic sequence of P_1 as an $R[x]$-module [1, Chap. XII, Section 1]. From this diagram we immediately deduce the exactness of the sequence

$$0 \longrightarrow P_1[x] \xrightarrow[(x-\alpha,\beta)]{} P_1[x] \oplus F_0[x] \xrightarrow[(j,-x^n)]{} P \longrightarrow 0. \tag{3}$$

This is the required presentation for P. If F_0 and P_1 are free, it reduces to the matrix presentation for $(xI - A \mid B)$ used in [5]. Since P is projective, (3) splits and we can reduce it mod x getting a split exact sequence

$$0 \longrightarrow P_1 \xrightarrow[(-\alpha,\beta)]{} P_1 \oplus F_0 \longrightarrow P/xP \longrightarrow 0. \tag{4}$$

Let $(\gamma, \delta): P_1 \oplus F_0 \to P_1$ be a splitting for (4), i.e., $-\gamma\alpha + \delta\beta = 1_{P_1}$.

LEMMA 1.3. *If γ is nilpotent, then $P \approx F$.*

Proof. Let $Q = \ker(\gamma, \delta)$ so $0 \to Q \to P_1 \oplus F_0 \xrightarrow[(\gamma,\delta)]{} P_1 \to 0$. Tensoring with $R[x]$ gives

$$0 \longrightarrow Q[x] \longrightarrow P_1[x] \oplus F_0[x] \xrightarrow[\gamma,\delta]{} P_1[x] \longrightarrow 0. \tag{5}$$

Consider the diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dl] \\
& P_1[x] \arrow[dr, "(x-\alpha,\beta)"] \arrow[dd, "\theta"'] & & Q[x] \arrow[dl] & \\
& & P_1[x] \oplus F_0[x] \arrow[dl, "(\gamma,\delta)"] \arrow[dr] & & \\
& P_1[x] \arrow[dl] & & P \arrow[dr] & \\
0 & & & & 0
\end{tikzcd} \tag{6}$$

where the two sequences in (6) are (3) and (5). Here $\theta = \gamma(x - \alpha) + \delta\beta = 1 + \gamma x$. If γ is nilpotent, θ is an isomorphism and the X-Lemma [3, Chap. I, Cor. 16.8] [9, Lemma 5.5] shows that $P \approx Q[x]$. Now since $x^n F \subset P \subset F$, we have $R[x, x^{-1}] \otimes_{R[x]} P \approx R[x, x^{-1}] \otimes_{R[x]} F$. Using $P \approx Q[x]$ and $F = F_0[x]$, this becomes $R[x, x^{-1}] \otimes_R Q \approx R[x, x^{-1}] \otimes_R F_0$. Factoring out $x - 1$ we get $Q \approx F_0$ so $P \approx Q[x] \approx F$.

LEMMA 1.4. *Suppose $-\gamma\alpha + \delta\beta = 1_{P_1}$ with $\alpha^2 = 0$ and suppose γ satisfies an equation $\gamma^2 + a\gamma + b = 0$ with $a, b \in R$. Then we can find (γ^*, δ^*): $P_1 \oplus F_0 \to P_1$ such that $-\gamma^*\alpha + \delta^*\beta = 1$ and $\gamma^{*2} = 0$.*

Proof. Set $\gamma^* = -\gamma\alpha(\gamma + a)$ and $\delta^* = (1 - \gamma\alpha)\delta$. Then

$$\gamma^{*2} = \gamma\alpha(\gamma + a)\,\gamma\alpha(\gamma + a) = \gamma\alpha(-b)\,\alpha(\gamma + a) = -b\gamma\alpha^2(\gamma + a) = 0.$$

Now $-\gamma^*\alpha = \gamma\alpha(\gamma + a)\alpha = (\gamma\alpha)^2 + a\gamma\alpha^2 = (\gamma\alpha)^2$ and $\delta^*\beta = (1 - \gamma\alpha)\,\delta\beta = (1 - \gamma\alpha)(1 + \gamma\alpha) = 1 - (\gamma\alpha)^2$ so $-\gamma^*\alpha + \delta^*\beta = 1$.

LEMMA 1.5. *Let M be a finitely generated projective R-module of* rank $\leqslant r$. *If $\gamma\colon M \to M$ then γ satisfies an equation $\gamma^r + a_1\gamma^{r-1} + \cdots + a_r = 0$ with $a_i \in R$.*

Proof. Let $M \oplus N \approx R^t$ be free and finitely generated. Let $f(X) = X^t + a_1 X^{t-1} + \cdots$ be the characteristic polynomial of $\gamma \oplus 0: M \oplus N \to M \oplus N$. By checking locally where M and N are free we see that $f(X) = X^{t-r} g(X)$ and $g(\gamma) = 0$.

Now, in the situation of Theorem 1.2, $P_1 \approx P/x^2F$ has rank $\leqslant 2$. Lemma 1.5 shows that Lemma 1.4 applies and Lemma 1.3 gives the required result.

2. Unimodular Rows

We will now show how the result on unimodular rows was derived from Theorem 3. The details will be omitted since we shall also give a direct proof of this result. This proof arose from a preliminary attempt to disprove Theorem 1.1 in the case where R is a unique factorization domain (UFD). Suppose we have $x^2F \subset P \subset F$ as in Theorem 1.1 with F free on 2 generators. Excluding the trivial cases considered in the proof of Theorem 1.1 we can assume that $M = F/P$ has rank 2 over R. Let $\theta: M \to M$ be multiplication by x so that $\theta^2 = 0$ and we can assume $\theta \neq 0$, otherwise $xF \subset P \subset F$. Suppose now that R is a UFD. If M is free on 2 generators, it is easy to see that θ is given by a matrix of the form

$$a \begin{pmatrix} bc & b^2 \\ -c^2 & -bc \end{pmatrix}$$

with $\gcd(b, c) = 1$. In general, M will be projective of rank 2 but by applying this result locally we can show that $\ker \theta$ is projective of rank 1 so $\ker \theta \approx R$ (since $\operatorname{Pic} R = 0$ for a UFD) and $\operatorname{Im} \theta$ is torsion free of rank 1 and hence isomorphic to an ideal of R. Let $s, t \in M$ be the images of a base for F and let $\ker \theta = Ru$. Then it is easy to see that M is presented by a single relation $-\beta t + \gamma s = \alpha u$, where (α, β, γ) is a unimodular row. Also $xs = b\beta u$, $xt = b\gamma u$, and $(\alpha, b\beta, b\gamma)$ is unimodular since s and t generate M as an $R[x]$-module. We can now drop the assumption that R is a UFD and simply assume that M is as just described. It is now a straightforward calculation to find a presentation for $P = \ker[F \to M]$. Then by a tedious and not very enlightening calculation with generators and relations, it can be shown that P is presented by 3 generators A, B, C with one relation $\alpha^2 C = -(\gamma + \alpha q x)A + (\beta - \alpha r x)B$. Here, if $F = R^2$ we have written $A = (b\beta\gamma - \alpha x, -b\beta^2)$, $B = (b\gamma^2, -b\beta\gamma - \alpha x)$, $C = (px\gamma + qx^2, -px\beta + rx^2)$, where $p\alpha + qb\beta + rb\gamma = 1$.

In other words, P is presented by the unimodular row $(\alpha^2, \beta - \alpha r x, \gamma + \alpha q x)$. Therefore, P/xP is presented by $(\alpha^2, \beta, \gamma)$. But P is free by Theorem 1.1. Therefore so is P/xP and our result follows by choosing $b = 1$. By going through the proof of Theorem 1.1 with this particular choice of P we get the following explicit version of the result.

THEOREM 2.1. *If* $p\alpha + q\beta + r\gamma = 1$, *then*

$$\begin{vmatrix} \alpha^2 & \beta & \gamma \\ \beta + r\alpha & -r^2 + pr\beta & -p + qr - pq\beta \\ \gamma - q\alpha & p + qr + pr\gamma & -q^2 - pq\gamma \end{vmatrix} = 1.$$

This can be verified by direct calculation. The following application was pointed out by A. Geramita.

COROLLARY 2.2. *Let k be any field and let $f(x, y) \in k[x, y]$. Let V be the affine variety defined by $z^n = f(x, y)$. If V is smooth and n is odd, then the tangent bundle of V is trivial.*

In fact, the tangent bundle of V is defined by the unimodular row $(\partial f/\partial x, \partial f/\partial y, -nz^{n-1})$. If $n = 0$ in k, it is easy to reduce this to $(0, 0, 1)$ by elementary transformations. If $n \neq 0$ in k, apply Theorem 2.1.

This proof does not extend to higher dimensions since Theorem 2.1 does not. However, in a number of interesting cases, the following result can be used to prove the triviality of the tangent bundle. The case where f is an isotropic ternary quadratic form over a field was done in [10, Prop. 5]. A. Geramita pointed out that the result is valid for quadratic forms in any number of variables over a field.

THEOREM 2.3. *Let R be a commutative ring with unit and let $f \in R[x_1, ..., x_n]$ be a homogeneous polynomial. Let $A = R[x_1, ..., x_n]/(f - 1)$ and let P be the projective A-module defined by the unimodular row $(x_1, ..., x_n)$. Suppose there is an element $(a_1, ..., a_n) \in R^n$ which is part of a free base for R^n such that $f(a_1, ..., a_n) = 0$. Then P is free.*

Proof. By making a linear change of the variables $x_1, ..., x_n$ over R, we can assume that $(a_1, ..., a_n) = (1, 0, ..., 0)$. Since $f(1, 0, ..., 0) = 0$, the coefficient of x_1^m in f is 0. Therefore $f = x_2 f_2 + \cdots + x_n f_n$ so $(x_2, ..., x_n)$ is unimodular. By elementary transformations, we can reduce $(x_1, x_2, ..., x_n)$ to $(1, x_2, ..., x_n)$ and then to $(1, 0, ..., 0)$.

COROLLARY 2.4. *Let f in Theorem 2.3 have degree m and assume that m is a unit in R. Then $(\partial f/\partial x_1, ..., \partial f/\partial x_n)$ is unimodular over A and defines a free projective module.*

Proof. By Euler's theorem $\sum x_i(\partial f/\partial x_i) = mf = m$ in A. Therefore $(\partial f/\partial x_1, ..., \partial f/\partial x_n)$ is unimodular and the projective module Q it defines is the dual of the module P in Theorem 2.3, i.e., $Q \approx \mathrm{Hom}_A(P, A)$.

3. Examples

We now give examples to show that Theorem 2.1 does not extend to longer rows. This does not mean that Theorem 1.1 becomes false for rank > 2 because the method of deducing Theorem 2.1 from Theorem 1.1 does not seem to generalize.

Let $R = \mathbb{C}[x_1, ..., x_{2n}]/(\sum x_i^2 - 1)$. Let $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4, ...,$ $z_n = x_{2n-1} + ix_{2n}$. Then $\sum z_i\bar{z}_i = 1$ so $(z_1, ..., z_n)$ is unimodular.

THEOREM 3.1. *If there is a prime p such that $\operatorname{ord}_p(m_1m_2 \cdots m_n) < \operatorname{ord}_p((n-1)!)$, then the projective module defined by the unimodular row $(z_1^{m_1}, ..., z_n^{m_n})$ is not free. In particular, this is so if there is a prime $p < n$ such that $p \nmid m_1m_2 \cdots m_n$.*

Proof. By [7], it is sufficient to prove the analogous result for complex bundles over S^{2n-1}. If E is the bundle on S^{2n-1} defined by $(z_1, ..., z_n)$, it is easy to see that the associated principal bundle is the canonical fibration $U(n-1) \to U(n) \to S^{2n-1}$. The bundle defined by $(z_1^{m_1}, ..., z_n^{m_n})$ is f^*E where $f: S^{2n-1} \to S^{2n-1}$ by

$$f(z_1, ..., z_n) = \frac{(z_1^{m_1}, ..., z_n^{m_n})}{(\sum |z_i|^{2m_i})^{1/2}}.$$

Let $U(n-1) \to P \to S^{2n-1}$ be the principal bundle associated with f^*E. Then we have

$$\begin{tikzcd} U(n-1) \arrow[r] \arrow[d, "="'] & P \arrow[r] \arrow[d] & S^{2n-1} \arrow[d, "f"] \\ U(n-1) \arrow[r] & U(n) \arrow[r] & S^{2n-1}. \end{tikzcd}$$

The associated ladder of homotopy groups gives

$$\begin{tikzcd} \pi_{2n-1}S^{2n-1} \arrow[r, "\partial_{f^*E}"] \arrow[d, "f_*"] & \pi_{2n-2}U(n-1) \arrow[r] \arrow[d, "="] & \pi_{2n-2}P \arrow[d] \\ \pi_{2n-1}S^{2n-1} \arrow[r, "\partial_E"'] & \pi_{2n-2}U(n-1) \arrow[r] & \pi_{2n-2}U(n). \end{tikzcd}$$

Now $\pi_{2n-1}S^{2n-1} = \mathbb{Z}$ and f_* is multiplication by $m_1m_2 \cdots m_n$ as is easily seen using the isomorphism $\pi_{2n-1}S^{2n-1} \approx H_{2n-1}(S^{2n-1})$. Also $\pi_{2n-2}U(n) = 0$ so ∂_E is onto and $\pi_{2n-2}U(n-1) = \mathbb{Z}/(n-1)!\,\mathbb{Z}$ [2]. It follows that the image of $\partial_E f_*$ has a nontrivial p-component so $\partial_E f_* \neq 0$ and hence $\partial_{f^*E} \neq 0$. Therefore $P \to S^{2n-1}$ is a nontrivial fibration.

In particular $(z_1^2, z_2, \ldots, z_n)$ is nontrivial for $n \geqslant 4$ since we can take $p = 3$. This shows that the obvious extension of Corollary 4 is false. It would be interesting to know just which $m_1, \ldots, m_n$ have the property that every unimodular row of the form $(a_1^{m_1}, a_2^{m_2}, \ldots, a_n^{m_n})$ represents a free module.

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